Wen Bin Guo, D. O. Revin, When is the search of relatively maximal subgroups reduced to quotient groups?, *Izvestiya: Mathematics*, 2022, Volume 86, Issue 6, 1102–1122

When is the search of relatively maximal subgroups reduced to quotient groups?

W. B. Guo and D. O. Revin

Abstract. Let $\mathfrak{X}$ be a class finite groups closed under taking subgroups, homomorphic images, and extensions, and let $\mathrm{k}_{\mathfrak{X}}(G)$ be the number of conjugacy classes $\mathfrak{X}$-maximal subgroups of a finite group G. The natural problem calling for a description, up to conjugacy, of the $\mathfrak{X}$-maximal subgroups of a given finite group is not inductive. In particular, generally speaking, the image of an $\mathfrak{X}$-maximal subgroup is not $\mathfrak{X}$-maximal in the image of a homomorphism. Nevertheless, there exist group homomorphisms that preserve the number of conjugacy classes of maximal $\mathfrak{X}$-subgroups (for example, the homomorphisms whose kernels are $\mathfrak{X}$-groups). Under such homomorphisms, the image of an $\mathfrak{X}$-maximal subgroup is always $\mathfrak{X}$-maximal, and, moreover, there is a natural bijection between the conjugacy classes of $\mathfrak{X}$-maximal subgroups of the image and preimage. In the present paper, all such homomorphisms are completely described. More precisely, it is shown that, for a homomorphism ϕ from a group G, the equality $\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{k}_{\mathfrak{X}}(\operatorname{im}\phi)$ holds if and only if $\mathrm{k}_{\mathfrak{X}}(\ker\phi) = 1$, which in turn is equivalent to the fact that the composition factors of the kernel of ϕ lie in an explicitly given list.

Keywords: finite group, complete class, $\mathfrak{X}$-maximal subgroup, Hall subgroup, reduction $\mathfrak{X}$-theorem.

§ 1. Introduction

1.1. The main result. In what follows, we will consider only finite groups, and a 'group' will always mean a 'finite group'.

A group from a class of groups $\mathfrak{X}$ will be simply called an $\mathfrak{X}$-*group*. The set of (inclusion) *maximal* $\mathfrak{X}$-*subgroups* (or $\mathfrak{X}$-*maximal subgroups*) of a group G will be denoted by $\mathrm{m}_{\mathfrak{X}}(G)$. The group G itself, acting on $\mathrm{m}_{\mathfrak{X}}(G)$ by conjugacies, splits this set into orbits (the conjugacy classes). The number of these classes is denoted by $\mathrm{k}_{\mathfrak{X}}(G)$. The term a '*relatively maximal subgroup*', which we used in the title of the present paper, was proposed by Wielandt [1] in order to denote $\mathfrak{X}$-maximal

W. Guo was supported by the National Natural Science Foundation of China (project no. 12171126), Wu Wen-Tsun Key Laboratory of Mathematics of Chinese Academy of Sciences and Key Laboratory of Engineering Modeling and Statistical Computation of Hainan Province. D. O. Revin was supported by RFBR and BRFBR, project no. 20-51-00007 and by the Russian Ministry for Education and Science within the framework of the state commission for the Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences (project no. FWNF-2022-0002).

AMS 2020 *Mathematics Subject Classification.* 20F28, 20D06, 20E22.

subgroups without indication of a concrete class $\mathfrak{X}$ and to distinguish them from maximal subgroups (in the usual sense, that is, from the maximal ones among the proper ones). Following Wielandt [2], [3], we say that the nonempty class $\mathfrak{X}$ of finite groups is *complete* if it is closed under taking subgroups, homomorphic images, and extensions. The last means that $G \in \mathfrak{X}$, whenever $N \in \mathfrak{X}$ and $G/N \in \mathfrak{X}$ for some normal subgroup N of the group G.

For a complete class $\mathfrak{X}$, the problem of *when is the search of $\mathfrak{X}$-maximal subgroups of a group G reduced to the analogous problem for the quotient group G/N*, as formulated in the title of the paper, turns out to be equivalent to the problem of *when* $\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{k}_{\mathfrak{X}}(G/N)$? The main result of the paper is the following

Theorem 1. *Let $\mathfrak{X}$ be a complete class of groups and N be a normal subgroup of a finite group G. Then if* $\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{k}_{\mathfrak{X}}(G/N)$, *then* $\mathrm{k}_{\mathfrak{X}}(N) = 1$.

The converse result to Theorem 1 was also proved in [4], Theorem 1. The following theorem holds.

Theorem 2. *Let $\mathfrak{X}$ be a complete class of groups and N be a normal subgroup of a finite group G. Then* $\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{k}_{\mathfrak{X}}(G/N)$ *if and only if* $\mathrm{k}_{\mathfrak{X}}(N) = 1$.

That is, the number of classes of conjugate $\mathfrak{X}$-maximal subgroups remains unchanged when transiting from a group G to the quotient group G/N if and only if in N all $\mathfrak{X}$-maximal subgroups are conjugate. There is also an exhaustive description of the groups A with $\mathrm{k}_{\mathfrak{X}}(A) = 1$: this condition is equivalent to saying that each nonabelian composition factor of the group A either lies in $\mathfrak{X}$, or is isomorphic to a simple group indicated in [4], Appendix A. So, in the case of a complete class $\mathfrak{X}$, we give a complete answer to the question raised in the title of the present paper.

The assumption in Theorem 2 that the class $\mathfrak{X}$ is complete is essential. Indeed, the conclusion of the theorem fails to hold if $\mathfrak{X} = \mathfrak{A}$ is the class of all abelian groups or $\mathfrak{X} = \mathfrak{N}$ is the class of all nilpotent groups. In both cases, $\mathfrak{X}$ is not closed under extensions and $\mathrm{k}_{\mathfrak{X}}(\mathrm{Sym}_3) = 2 \neq 1 = \mathrm{k}_{\mathfrak{X}}(\mathrm{Sym}_3 / \mathrm{Alt}_3)$, even though $\mathrm{k}_{\mathfrak{X}}(\mathrm{Alt}_3) = 1$.

1.2. Motivation and historical remarks. The following class of problems appears in the finite group theory and its applications starting from the seminal studies of É. Galois and C. Jordan: *given a group G (for example, a symmetric group) and a class $\mathfrak{X}$ of finite groups (for example, the class of solvable groups), find the $\mathfrak{X}$-subgroups of the group G.* It seems that problems of this kind cannot be successfully attacked in the general setting. If the class $\mathfrak{X}$ is complete (similarly to the class of solvable groups), one may confine oneself with search of $\mathfrak{X}$-maximal subgroups.

In what follows, $\mathfrak{X}$ will always denote a fixed complete class. In addition to the class $\mathfrak{S}$ of solvable groups, among typical examples of complete classes we mention the class $\mathfrak{G}_\pi$ of all π-groups and the class $\mathfrak{S}_\pi$ of all solvable π-groups for a given subset π of the set $\mathbb{P}$ of all primes (recall that a *π-group* is a group in which any prime divisor of the order lies in π). Note that, for the class $\mathfrak{X}$, we have the inclusions:

$$\mathfrak{S}_\pi \subseteq \mathfrak{X} \subseteq \mathfrak{G}_\pi,$$

here π is the set

$$\pi(\mathfrak{X}) = \{p \in \mathbb{P} \mid \text{ there exists } G \in \mathfrak{X} \text{ such that } p \text{ divides } |G|\}.$$

The classes of π-separable and π-solvable groups[1] are also complete.

It is natural that the $\mathfrak{X}$-maximal subgroups should be studied up to a conjugacy. By an $\mathfrak{X}$-*scheme* we will mean a complete system of representatives of its classes of conjugate $\mathfrak{X}$-maximal subgroups. The cardinality of an $\mathfrak{X}$-scheme of a group G is defined as the above number $\mathrm{k}_{\mathfrak{X}}(G)$. The main aim in the problems mentioned above can be looked upon as the search of an $\mathfrak{X}$-scheme and description of the structure of its elements.

If $\mathfrak{X} = \mathfrak{G}_p$ is the class of p-groups for a prime p, then any $\mathfrak{X}$-maximal subgroup is a Sylow p-subgroup. Such subgroups in any group are conjugate [5]. It is also worth mentioning that the $\mathfrak{X}$-maximal subgroups of solvable groups are, precisely, the so-called $\pi(\mathfrak{X})$-Hall subgroups, which form a conjugacy class by Hall's theorem [6]. The search of the Sylow and Hall subgroups is substantially facilitated by their properties that allow one to change from a group to sections of a normal or a subnormal series in inductive arguments. For example, if H is a Sylow p-subgroup of the group G and $N \trianglelefteq G$, then $H \cap N$ and HN/N are Sylow p-subgroups in N and G/N, respectively.

In the general case, the problems under considerations are highly noniductive, inasmuch as both the intersection $H \cap N$ of the subgroups $H \in \mathrm{m}_{\mathfrak{X}}(G)$ and $N \trianglelefteq G$, or the image of HN/N in G/N (or, equivalently, the image of H under an arbitrary epimorphism) may fail to be $\mathfrak{X}$-maximal subgroups in N and G/N (see [2], [3]). However, for intersections with normal subgroups, the situation can be partially improved by studying the $\mathfrak{X}$-submaximal subgroups,[2] which are generalizations of $\mathfrak{X}$-maximal subgroups (see [3]).

The present paper is concerned with the behaviour of $\mathfrak{X}$-maximal subgroups under homomorphisms. It is known (see [2], § 14.2, [3], § 4.3) that if, for a class $\mathfrak{X}$, there exists a group L with nonconjugate $\mathfrak{X}$-maximal subgroups, then any group G is the image of a homomorphism (more precisely, of the natural epimorphism from the regular wreath product $L \wr G$) under which *each* (not only $\mathfrak{X}$-maximal) $\mathfrak{X}$-subgroup coincides with the image of some $\mathfrak{X}$-maximal subgroup. In other words, an attempt to extend the concept of an $\mathfrak{X}$-maximal subgroup to be in accord with homomorphic images calls for the study of all $\mathfrak{X}$-subgroups. Another challenge associated with the transition to the epimorphic image is that the images of nonconjugate $\mathfrak{X}$-maximal subgroups may happen to be conjugate, and, as a result, information on conjugacy classes may be lost.

Consequently, it is important to describe all the cases where a transition from a group G to the quotient group G/N is a reduction for the highlighted type of problems, that is, when $\mathfrak{X}$-maximality of subgroups is preserved and information on their conjugacy is not distorted. In other words, it is important to know when

[1] Recall that a group is called π-*separable* if it admits a (sub)normal series in which all factors are π- or π'-groups, where $\pi' = \mathbb{P} \setminus \pi$. If, in addition, all π-factors of this series are solvable, the group is called π-*solvable*.

[2] According to Wielandt [3], a subgroup H of a group G is called an $\mathfrak{X}$-*submaximal* if G can be embedded as a subnormal subgroup into some group G^* so that $H = H^* \cap G$ for an appropriate $H^* \in \mathrm{m}_{\mathfrak{X}}(G^*)$.

an $\mathfrak{X}$-scheme is carried over to an $\mathfrak{X}$-scheme; in particular,

$$\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{k}_{\mathfrak{X}}(G/N). \tag{1.1}$$

The situation $N \in \mathfrak{X}$ is an example. Less straightforward examples were given by Wielandt in [2], who, following Chunikhin [7]–[10], proposed a programme for finding all 'good' cases. The present paper concludes this program.

Equality (1.1) is not only necessary but also a sufficient condition that a canonical (or any other) epimorphism $\overline{}: G \to G/N$ would send an $\mathfrak{X}$-scheme of a group G into the $\mathfrak{X}$-scheme G/N. Indeed, it is known that each $\mathfrak{X}$-subgroup of $\overline{G}$ is the image of an $\mathfrak{X}$-subgroup from G (see Lemma 1). Hence $\mathrm{m}_{\mathfrak{X}}(\overline{G}) \subseteq \{\overline{H} \mid H \in \mathrm{m}_{\mathfrak{X}}(G)\}$ and $\mathrm{k}_{\mathfrak{X}}(\overline{G}) \leqslant \mathrm{k}_{\mathfrak{X}}(G)$. So, (1.1) implies

$$\mathrm{m}_{\mathfrak{X}}(\overline{G}) = \{\overline{H} \mid H \in \mathrm{m}_{\mathfrak{X}}(G)\}.$$

Therefore, the existence of *some* one-one correspondence between classes of conjugate $\mathfrak{X}$-maximal subgroups in the groups G and $\overline{G} = G/N$, as implied by (1.1), gives evidence about the presence of a *natural* one-one correspondence between these classes induced by the mapping $H \mapsto \overline{H}$.

We will say that the *reduction $\mathfrak{X}$-theorem*

– *holds for a pair* (G, N), $N \trianglelefteq G$, if (1.1) holds;

– *holds for a group* A if it holds for any pair (G, N) with $N \cong A$.

Setting $G = A$, we see that the reduction $\mathfrak{X}$-theorem for a group A implies the conjugacy of the $\mathfrak{X}$-maximal subgroups: $\mathrm{k}_{\mathfrak{X}}(A) = \mathrm{k}_{\mathfrak{X}}(A/A) = 1$. Wielandt [2], § 15.4, noted that the reduction $\mathfrak{X}$-theorem itself for A, would, in turn, follow from the conjugacy of the $\mathfrak{X}$-*sub*maximal subgroups, and raised the conjecture (see [2], Offene Frage zu 15.4, which was proved later in [4], Theorem 1), to the effect that the conjugacy of the $\mathfrak{X}$-maximal and $\mathfrak{X}$-submaximal subgroups are equivalent. Therefore, the reduction $\mathfrak{X}$-theorem *for the group* A is equivalent to the equality $\mathrm{k}_{\mathfrak{X}}(A) = 1$. Next, the condition $\mathrm{k}_{\mathfrak{X}}(A) = 1$ is equivalent to saying that $\mathrm{k}_{\mathfrak{X}}(S) = 1$ for any composition factor S for the group A. If S is a simple group, then necessary and sufficient arithmetic conditions on natural parameters[3] of the group S for the equality $\mathrm{k}_{\mathfrak{X}}(S) = 1$ to hold are known (see [4], Theorem 1, Appendix A). So, the results of [4] can be interpreted as a description of all such pairs (G, N) for which equality (1.1) is controlled only by the isomorphism type of the group N.

The isomorphism type of a group G and of its normal subgroup N do not define uniquely the number $\mathrm{k}_{\mathfrak{X}}(G/N)$. For example, the group $G = \mathrm{PSL}_2(7) \times \mathrm{PGL}_2(7)$ has two normal subgroups N_1 and N_2 such that $N_1 \cong N_2 \cong \mathrm{PSL}_2(7)$, $G/N_1 \cong \mathbb{Z}_2 \times \mathrm{PSL}_2(7)$, and $G/N_2 \cong \mathrm{PGL}_2(7)$. However, for the class $\mathfrak{X} = \mathfrak{S}$ of solvable groups it can be easily shown (see, for example, [11]) that $\mathrm{k}_{\mathfrak{X}}(G/N_1) = 3$, while $\mathrm{k}_{\mathfrak{X}}(G/N_2) = 4$.

Nevertheless, in view of Theorem 1, the equality $\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{k}_{\mathfrak{X}}(G/N)$ for a group G and its normal subgroup N is an intrinsic property of the group N, which does not depend only on the particularities of the embedding of N into G, but also on the group G itself, and which implies the reduction $\mathfrak{X}$-theorem for N. This is the fact

[3]For example, for the group $S = \mathrm{PSL}_n(q)$, where q is a power of $p \in \pi(\mathfrak{X})$, the equality $\mathrm{k}_{\mathfrak{X}}(S) = 1$ is equivalent to saying that either $S \in \mathfrak{X}$, or $s \mid |S| \Rightarrow s \mid q(q-1)$ and $s > n$ for all $s \in \pi(\mathfrak{X})$.

which advances Theorem 1 in comparison with Theorem 1 in [4]. It is worth pointing out that whereas Theorem 1 in [4] was proved by reducing the general situation to the known particular case $\mathfrak{X} = \mathfrak{G}_\pi$ (see [13], [12]), our result in Theorem 1 was unknown even for this case.

From the description of all *groups* for which the reduction $\mathfrak{X}$-theorem (see [4], Theorem 1) holds, we get a description of all *pairs* for which it holds.

Corollary 1. *Let $\mathfrak{X}$ be a complete class. Then a necessary and sufficient condition that the reduction $\mathfrak{X}$-theorem hold for a pair (G, N) is that, for any composition factor S of the group N, either $S \in \mathfrak{X}$, or one of conditions* I–VII *in* [4], *Appendix A, is met for the pair $(S, \mathfrak{X})$.*

1.3. Some corollaries. Since $\mathrm{k}_\mathfrak{X}(G/N) \leqslant \mathrm{k}_\mathfrak{X}(G)$ for any normal subgroup N of a group G, the following result is a direct consequence of Theorem 1.

Corollary 2. *Let $\mathfrak{X}$ be a complete class of groups and N be a normal subgroup of a finite group G such that $\mathrm{k}_\mathfrak{X}(N) > 1$. Then $\mathrm{k}_\mathfrak{X}(G) > \mathrm{k}_\mathfrak{X}(G/N)$.*

Moreover, as the above example of the group $G = \mathrm{PSL}_2(7) \times \mathrm{PGL}_2(7)$ shows, the precise value $\mathrm{k}_\mathfrak{X}(G/N)$ does not only depend on the numbers $\mathrm{k}_\mathfrak{X}(G)$ and $\mathrm{k}_\mathfrak{X}(N)$ themselves, but is not even controlled by the isomorphism type of the groups G and N.

From Theorem 2 it follows that any group G contains the largest normal subgroup R such that $\mathrm{k}_\mathfrak{X}(G) = \mathrm{k}_\mathfrak{X}(G/R)$.

Corollary 3. *Let $\mathfrak{X}$ be a complete class. For an arbitrary finite group G, consider the subgroup $R = \langle N \mid N \trianglelefteq G$ and $\mathrm{k}_\mathfrak{X}(G) = \mathrm{k}_\mathfrak{X}(G/N)\rangle$. This subgroup has the following properties*:

(i) $\mathrm{k}_\mathfrak{X}(G) = \mathrm{k}_\mathfrak{X}(G/R)$;

(ii) *if* $N \trianglelefteq G$ *and* $N \leqslant R$, *then* $\mathrm{k}_\mathfrak{X}(G) = \mathrm{k}_\mathfrak{X}(G/N)$;

(iii) *if* $\overline{G} = G/R$, *then* $\mathrm{k}_\mathfrak{X}(\overline{G}) = \mathrm{k}_\mathfrak{X}(\overline{G}/\overline{N})$ *implies* $\overline{N} = 1$ *for any* $\overline{N} \trianglelefteq \overline{G}$.

In view of Theorem 2, the subgroup $R \leqslant G$ from Corollary 3 coincides with the $\mathscr{D}_\mathfrak{X}$-radical of the group G, where, as in [14], [4], by $\mathscr{D}_\mathfrak{X}$ we denote the class of finite groups in which all $\mathfrak{X}$-maximal subgroups are conjugate. The class $\mathscr{D}_\mathfrak{X}$ is closed under taking normal subgroups of homomorphic images and extensions,[4] and, in particular, is a Fitting class (see the definition in [15]), and hence in any group there exits the $\mathscr{D}_\mathfrak{X}$-radical. Note that, in general, $\mathscr{D}_\mathfrak{X}$ is not a complete class, because it may fail to be closed under taking subgroups (see [16], Theorem 1.7).

The quotient group G/R will be called a *complete reduction over* $\mathfrak{X}$ *of a group* G, and the subgroup $R = \langle N \mid N \trianglelefteq G$ and $\mathrm{k}_\mathfrak{X}(G) = \mathrm{k}_\mathfrak{X}(G/N)\rangle$ itself will be called the *kernel of reduction.* A group G will be called *completely reduced over* $\mathfrak{X}$ if the kernel of its reduction is trivial. The search problem of an $\mathfrak{X}$-scheme can be reduced to consideration of completely reduced groups.

Given a group G, let

$$\mathrm{om}_\mathfrak{X}(G) = \{K \leqslant G \mid \mathrm{m}_\mathfrak{X}(K) \cap \mathrm{m}_\mathfrak{X}(G) \neq \varnothing\}$$

[4]See Corollary 1 in [4]. Note also that in view of the inequality $\mathrm{k}_\mathfrak{X}(G/N) \leqslant \mathrm{k}_\mathfrak{X}(G)$ this result is a particular case of Theorem 2.

be the set of all overgroups of $\mathfrak{X}$-maximal subgroups. The following result, which follows from Theorem 2 and the main result of [16], is not so obvious.

Corollary 4. *Let $\mathfrak{X}$ be a complete class of groups and let N be a normal subgroup of a finite group G. Then the following conditions are equivalent*:

(i) $\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{k}_{\mathfrak{X}}(G/N)$;

(ii) $\mathrm{k}_{\mathfrak{X}}(K) = \mathrm{k}_{\mathfrak{X}}(K/(K \cap N))$ *for all* $K \in \mathrm{om}_{\mathfrak{X}}(G)$.

1.4. The category of groups and $\mathfrak{X}$-isoschematisms. Let us formulate Theorem 2 in the language of homomorphisms. An epimorphism $\phi\colon G \to G^*$ will be said to be an *isoschematism over* $\mathfrak{X}$ (or, simply, an $\mathfrak{X}$-*isoschematism*) if it maps an $\mathfrak{X}$-scheme of the group G (each or some) into an $\mathfrak{X}$-scheme of the group G^*. Theorem 2 is equivalent to saying that *an epimorphism ϕ is an $\mathfrak{X}$-isoschematism if and only if* $\mathrm{k}_{\mathfrak{X}}(\ker\phi) = 1$.

According to the above, $\mathfrak{X}$-isoschematicity of an epimorphism $\phi\colon G \to G^*$ is completely determined only by the groups G and G^*, and is independent of a concrete mapping. In other words, the following result holds.

Proposition 1. *Let G be a finite group, and let G^* be its epimorphic image. Given a complete class $\mathfrak{X}$, the following assertions are equivalent*:

(i) *there exists an $\mathfrak{X}$-isoschematism* $\phi\colon G \to G^*$;

(ii) *any epimorphism $\phi\colon G \to G^*$ is an $\mathfrak{X}$-isoschematism*;

(iii) $\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{k}_{\mathfrak{X}}(G^*)$.

The kernels of two $\mathfrak{X}$-isoschematisms from G onto G^* may fail to be isomorphic, even though they have the same set of composition factors by the Jordan–Hölder theorem.

Existence of an $\mathfrak{X}$-isoschematism from G onto G^* is written as

$$G \underset{\mathfrak{X}}{\twoheadrightarrow} G^*.$$

The same symbol will also be used in the notation

$$\phi\colon G \underset{\mathfrak{X}}{\twoheadrightarrow} G^*,$$

which means that the mapping ϕ is an $\mathfrak{X}$-isoschematism from G onto G^*.

The relation $\underset{\mathfrak{X}}{\twoheadrightarrow}$ can be considered as a relation between groups. Clearly, this relation is reflexive and transitive, but not symmetric. Let us symmetrize it. We say that two groups G_1 and G_2 are *isoschemic over* $\mathfrak{X}$ (or $\mathfrak{X}$-*isoschemic*), written

$$G_1 \underset{\mathfrak{X}}{\equiv} G_2,$$

if there exist $\mathfrak{X}$-isoschematisms from G_1 and G_2 onto the same group:

$$\begin{array}{ccc} G_1 & \underset{\mathfrak{X}}{\equiv} & G_2 \\ & {\scriptstyle\mathfrak{X}}\searrow \quad \swarrow{\scriptstyle\mathfrak{X}} & \\ & G_0. & \end{array}$$

The relation $\underset{\mathfrak{X}}{\equiv}$ is, clearly, reflexive and symmetric. In the actual fact, this relation defines an equivalence on groups; its transitivity follows from Theorem 2. Using this relation, one can describe the category of groups and $\mathfrak{X}$-isoschematisms.

Corollary 5. *For finite groups G_1 and G_2, $G_1 \underset{\mathfrak{X}}{\equiv} G_2$ if and only if the complete reductions of these groups over $\mathfrak{X}$ are isomorphic. The relation $\underset{\mathfrak{X}}{\equiv}$ is an equivalence on finite groups. Each equivalence class is a subcategory in the category of all groups and $\mathfrak{X}$-isoschematisms and contains a unique* (*up to isomorphism*) *completely reduced over $\mathfrak{X}$ group which is an universally attracting object*[5] *in this subcategory.*

In the language of homomorphisms, Corollary 4 can be stated as follows. *Let ϕ be an $\mathfrak{X}$-isoschematism defined on a group G, and let K be an overgroup of an $\mathfrak{X}$-maximal subgroup from G. Then the restriction of ϕ to K is an $\mathfrak{X}$-isoschematism $K \underset{\mathfrak{X}}{\twoheadrightarrow} K^{\phi}$.*

§ 2. Notation and preliminary lemmas

We will use the following standard notation from the group theory (see, for example, [11], [15], [18]–[20]). Given a natural number n, by $\pi(n)$ we denote the set of its prime divisors; for a group G, we set $\pi(G) = \pi(|G|)$. For a fixed set $\pi \subseteq \mathbb{P}$ of primes and a complete class $\mathfrak{X}$ of finite groups, we will use the following not quite standard notation:

Ω/G is the set of orbits of an action of the group G on a set Ω;

$|\Omega : G|$ is the number of orbits of an action of G on Ω, i.e. $|\Omega : G| = |\Omega/G|$

$\mathrm{Hall}_{\mathfrak{X}}(G)$ is the set of $\mathfrak{X}$-*Hall subgroups* of the group G — these being the $\mathfrak{X}$-subgroups whose index is not divisible by any number from $\pi(\mathfrak{X})$;

$\mathrm{Hall}_{\pi}(G)$ is the set of π-Hall subgroups in G, that is, $\mathrm{Hall}_{\mathfrak{X}}(G)$ for $\mathfrak{X} = \mathfrak{G}_{\pi}$;

$\mathrm{m}_{\mathfrak{X}}(G)$ is the set of $\mathfrak{X}$-*maximal subgroups* of the group G;

$\mathrm{k}_{\mathfrak{X}}(G)$ is the number of the conjugacy classes of $\mathfrak{X}$-maximal subgroups of the group G, that is, $\mathrm{k}_{\mathfrak{X}}(G) = |\mathrm{m}_{\mathfrak{X}}(G) : G|$ for the action of the group G by conjugations on the set $\mathrm{m}_{\mathfrak{X}}(G)$;

$\mathrm{h}_{\mathfrak{X}}(G)$ is the number of conjugacy classes of $\mathfrak{X}$-Hall subgroups of the group G, that is, $\mathrm{h}_{\mathfrak{X}}(G) = |\mathrm{Hall}_{\mathfrak{X}}(G) : G|$ for the action of the group G by conjugations on the set $\mathrm{Hall}_{\mathfrak{X}}(G)$;

$\mathscr{E}_{\mathfrak{X}}$ is the class of all finite groups G such that $\mathrm{h}_{\mathfrak{X}}(G) \geqslant 1$ (or, equivalently, $\mathrm{Hall}_{\mathfrak{X}}(G) \neq \varnothing$);

$\mathscr{C}_{\mathfrak{X}}$ is the class of all finite groups G such that $\mathrm{h}_{\mathfrak{X}}(G) = 1$;

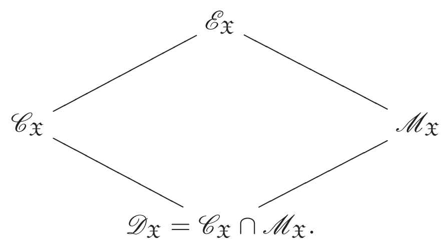
$\mathscr{D}_{\mathfrak{X}}$ is the class of all finite groups G such that $\mathrm{k}_{\mathfrak{X}}(G) = 1$;

$\mathscr{M}_{\mathfrak{X}}$ is the class of all finite groups G such that $\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{h}_{\mathfrak{X}}(G)$ (or, equivalently, $\mathrm{m}_{\mathfrak{X}}(G) = \mathrm{Hall}_{\mathfrak{X}}(G)$).

The notation $\mathscr{E}_{\mathfrak{X}}$, $\mathscr{C}_{\mathfrak{X}}$ and $\mathscr{D}_{\mathfrak{X}}$, which generalizes P. Hall's notation $\mathscr{E}_{\pi}$, $\mathscr{C}_{\pi}$ and $\mathscr{D}_{\pi}$ (see [21], and also [15], Ch. I, § 3, [20], Ch. 5, § 3), is equivalent to that used by P. Hall if $\mathfrak{X} = \mathfrak{G}_{\pi}$ is the class of all π-groups. By definition, $\mathscr{D}_{\mathfrak{X}} = \mathscr{C}_{\mathfrak{X}} \cap \mathscr{M}_{\mathfrak{X}}$. The

[5] See the definition in [17], Ch. 1, § 7. Note that in this category, $\mathfrak{X}$-isoschematisms are considered as morphisms up to a composition with automorphisms of groups.

inclusions between the classes $\mathscr{E}_{\mathfrak{X}}$, $\mathscr{C}_{\mathfrak{X}}$, $\mathscr{M}_{\mathfrak{X}}$ and $\mathscr{D}_{\mathfrak{X}}$ are shown in the diagram



In the case $\mathfrak{X} = \mathfrak{G}_\pi$, we will use the natural notation $\mathrm{k}_\pi(G)$ and $\mathrm{h}_\pi(G)$, respectively, for the number of conjugacy classes of π-maximal and π-Hall subgroups of a group G. We will say that $n \in \mathbb{N}$ is a *π-number* if $\pi(n) \subseteq \pi$.

Lemma 1 (see [22], Ch. III, Theorem 3.9). *Let $\mathfrak{X}$ be a complete class. Given a group homomorphism $\phi\colon G \to G_0$, suppose that $K \in \mathfrak{X}$ for some subgroup $K \leqslant G^\phi$. Then $K = H^\phi$ for some $\mathfrak{X}$-subgroup $H \leqslant G$. In particular,* $\mathrm{m}_{\mathfrak{X}}(G^\phi) \subseteq \mathrm{m}_{\mathfrak{X}}(G)^\phi$.

By $\mathfrak{X}'$ we denote the class of all groups G such that $\mathrm{m}_{\mathfrak{X}}(G) = \{1\}$. A group is called $\mathfrak{X}$-*separable* if it admits a (sub)normal series in which each factor is either an $\mathfrak{X}$- or an $\mathfrak{X}'$-group.

The following lemma summarizes some known results on the behaviour of $\mathfrak{X}$-maximal and $\mathfrak{X}$-Hall subgroups.

Lemma 2. *Let N be a normal subgroup of the group G. Then the following assertions hold.*

(i) *If $H \in \mathrm{Hall}_{\mathfrak{X}}(G)$, then $H \cap N \in \mathrm{Hall}_{\mathfrak{X}}(N)$ and $HN/N \in \mathrm{Hall}_{\mathfrak{X}}(G/N)$ (see* [20], *Ch.* IV, § 5.11).

(ii) *Let $G/N \in \mathfrak{X}$. Then a necessary and sufficient condition that, for $H \in \mathrm{Hall}_{\mathfrak{X}}(N)$, there exist a subgroup $K \in \mathrm{Hall}_{\mathfrak{X}}(G)$ such that $H = K \cap N$ is that $H^N = H^G$ (that is, when the class $H^N \in \mathrm{Hall}_{\mathfrak{X}}(N)/N$ is invariant under the action of the group G on the set $\mathrm{Hall}_{\mathfrak{X}}(N)/N$; see* [23], *Lemma* 2.1, (*e*)).

(iii) *Let N be an $\mathfrak{X}$-separable group. Then $\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{k}_{\mathfrak{X}}(G/N)$. In particular, $G \in \mathscr{D}_{\mathfrak{X}}$ if and only if $G/N \in \mathscr{D}_{\mathfrak{X}}$ (see* [2], § 12.9).

Lemma 3 (see [4], Theorem 1). *Let N be a normal subgroup of a group G and $\mathrm{k}_{\mathfrak{X}}(N) = 1$. Then $\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{k}_{\mathfrak{X}}(G/N)$.*

Given subgroups S, H of the group G, we denote by $\mathrm{Aut}_H(S)$ the *H-induced automorphism group of the group S*, that is, the image in $\mathrm{Aut}(S)$ of the homomorphism

$$\alpha_H : \mathrm{N}_H(S) \to \mathrm{Aut}(S),$$

which associates with each $x \in \mathrm{N}_H(S)$ the automorphism of the group S defined by $s \mapsto s^x = x^{-1}sx$ for all $s \in S$. The kernel of this homomorphism is $\mathrm{C}_H(S)$, and hence

$$\mathrm{Aut}_H(S) \cong \mathrm{N}_H(S)/\mathrm{C}_H(S).$$

If $H \leqslant K \leqslant G$, then the homomorphism α_H is the restriction of the homomorphism $\alpha_K : \mathrm{N}_K(S) \to \mathrm{Aut}(S)$ to $\mathrm{N}_H(S)$. Therefore, $\mathrm{Aut}_H(S) \leqslant \mathrm{Aut}_K(S)$.

Lemma 4. *Let S be a simple nonabelian subnormal subgroup of a group G, let $H \in \mathrm{Hall}_{\mathfrak{X}}(S)$, and let the group $\mathrm{Aut}_G(S)$ stabilize the conjugacy class of the subgroup H in S (that is, $H^{\mathrm{Aut}_G(S)} = H^S$). Consider an arbitrary right transversal $g_1, \dots, g_n$ of the subgroup $\mathrm{N}_G(S)$ in G. Let*

$$M = \langle S^{g_i} \mid i = 1, \dots, n\rangle \quad \textit{and} \quad V = \langle H^{g_i} \mid i = 1, \dots, n\rangle.$$

Then $M = \langle S^G\rangle$ is a minimal normal subgroup of the group G, and
(i) $V^G = V^M$;
(ii) $V \in \mathrm{Hall}_{\mathfrak{X}}(M)$;
(iii) *if $G/M \in \mathfrak{X}$, then $V = K \cap M$ and $H = K \cap S$ for some $K \in \mathrm{Hall}_{\mathfrak{X}}(G)$.*

Proof. For any $g \in G$ we have $S^g \in \{S^{g_i} \mid i = 1, \dots, n\}$, and hence $M = \langle S^G\rangle \trianglelefteq G$. We also note that $[S^{g_i}, S^{g_j}] = 1$ for $i \neq j$ since the subgroup S is simple and subnormal.

Let $g \in G$. There exist a permutation $\sigma \in \mathrm{Sym}_n$ and elements $x_1, \dots, x_n \in \mathrm{N}_G(S)$ such that $g_i g = x_i g_{i\sigma}$. Consider the automorphisms $\gamma_i \in \mathrm{Aut}_G(S)$ defined by $\gamma_i \colon s \mapsto s^{x_i}$. By the assumption, $H^{x_i} = H^{\gamma_i} = H^{s_i}$ for some $s_i \in S$. We set $a_i = s_{i\sigma^{-1}}^{g_i}$ and $a = a_1 \cdots a_n$. It is clear that $a \in M$. The equality $V^G = V^M$ will be verified if show that $V^g = V^a$.

By definition $a_i \in S^{g_i}$ and $H^{g_i a} = H^{g_i a_i}$. We have

$$\begin{aligned} V^g &= \langle H^{g_i g} \mid i = 1, \dots, n\rangle = \langle H^{x_i g_{i\sigma}} \mid i = 1, \dots, n\rangle \\ &= \langle H^{s_i g_{i\sigma}} \mid i = 1, \dots, n\rangle = \langle H^{s_{i\sigma^{-1}} g_i} \mid i = 1, \dots, n\rangle \\ &= \langle H^{g_i s_{i\sigma^{-1}}^{g_i}} \mid i = 1, \dots, n\rangle = \langle H^{g_i a_i} \mid i = 1, \dots, n\rangle = \langle H^{g_i a} \mid i = 1, \dots, n\rangle = V^a. \end{aligned}$$

This proves assertion (i). Next, V is a direct product of the $\mathfrak{X}$-groups H^{g_i}, $i = 1, \dots, n$. Hence $V \in \mathfrak{X}$. Further, since $H \in \mathrm{Hall}_{\mathfrak{X}}(S)$, the number

$$|M : V| = \prod_{i=1}^{n} |S^{g_i} : H^{g_i}| = |S : H|^n$$

is not divisible by any number from $\pi(\mathfrak{X})$. Hence $V \in \mathrm{Hall}_{\mathfrak{X}}(M)$, which proves (ii). Finally, (iii) is secured by (i) and Lemma 2, (ii). Lemma 4 is proved.

Lemma 5. *Let a normal subgroup N of a group G be a direct product of nonabelian simple groups, and S be one of these groups. Suppose that $G = KN$ for some subgroup K. Then*
(i) $\mathrm{N}_G(S) = N\mathrm{N}_K(S)$;
(ii) $\mathrm{Aut}_G(S) = \mathrm{Inn}(S)\,\mathrm{Aut}_K(S)$.

Proof. Let $N = S_1 \times S_2 \times \dots \times S_n$ and $S_1 = S$. Then

$$N \leqslant \mathrm{N}_G(S) \text{ and } S_2 \times \dots \times S_n = \mathrm{C}_N(S) \leqslant \mathrm{C}_G(S).$$

Hence $\mathrm{N}_G(S) = N\mathrm{N}_K(S)$, as claimed in (i). Let

$$\alpha \colon \mathrm{N}_G(S) \to \mathrm{Aut}(S)$$

denote the natural homomorphism induced by conjugations. Its kernel is $\mathrm{C}_G(S)$. We have $S^\alpha = \mathrm{Aut}_S(S) = \mathrm{Inn}(S)$, $N = S\mathrm{C}_N(S)$. Hence $N^\alpha = \mathrm{Inn}(S)$ and

$$\mathrm{Aut}_G(S) = \mathrm{N}_G(S)^\alpha = N^\alpha \mathrm{N}_K(S)^\alpha = \mathrm{Inn}(S)\,\mathrm{Aut}_K(S).$$

This proves assertion (ii), and, therefore, the lemma.

The key role in the proof of Theorem 1 is played by the theorem on the number of classes of conjugate π-Hall subgroups in simple groups (see [23]). We will use the following refined version of this result.

Lemma 6 (see [23], Theorem 1.1). *Let S be a simple finite group possessing a π-Hall subgroup for some set π of primes. Then one of the following assertions holds*:

(i) $2 \notin \pi$ *and* $\mathrm{h}_\pi(S) = 1$;

(ii) $3 \notin \pi$ *and* $\mathrm{h}_\pi(S) \in \{1, 2\}$;

(iii) $2, 3 \in \pi$ *and* $\mathrm{h}_\pi(S) \in \{1, 2, 3, 4, 9\}$.

Lemma 7 (see [14], Lemma 12). *Let $\mathfrak{X}$ be a complete class. We set $\pi = \pi(\mathfrak{X})$. Suppose also that $\mathrm{h}_\pi(S) = 9$. Then $\mathrm{h}_\mathfrak{X}(S)$ is one of the numbers* 0, 1 *or* 9.

The following result is a consequence of Lemmas 6 and 7.

Lemma 8. *Let S be a simple finite group. Then one of the following assertions holds*:

(i) $2 \notin \pi(\mathfrak{X})$ *and* $\mathrm{h}_\mathfrak{X}(S) \in \{0, 1\}$;

(ii) $3 \notin \pi(\mathfrak{X})$ *and* $\mathrm{h}_\mathfrak{X}(S) \in \{0, 1, 2\}$;

(iii) $2, 3 \in \pi(\mathfrak{X})$ *and* $\mathrm{h}_\mathfrak{X}(S) \in \{0, 1, 2, 3, 4, 9\}$.

Assume that a simple group S satisfies $\mathrm{h}_\mathfrak{X}(S) = 9$. Since $\mathrm{h}_\mathfrak{X}(S) \leqslant \mathrm{h}_\pi(S)$ for $\pi = \pi(\mathfrak{X})$, by Lemma 6 we have $\mathrm{h}_\mathfrak{X}(S) = \mathrm{h}_\pi(S)$, and, therefore, $\mathrm{Hall}_\mathfrak{X}(S) = \mathrm{Hall}_\pi(S)$. Now, in view of Lemmas 2.3, 3.1, 4.4, 8.1 in [23], we get the following result on the structure of $\mathfrak{X}$-Hall subgroups of a group S.

Lemma 9. *Let S be a simple finite group and $\mathrm{h}_\mathfrak{X}(S) = 9$ for some complete class $\mathfrak{X}$. Then the following assertions hold.*

(i) $S \cong \mathrm{PSp}_{2n}(q) \cong \mathrm{PSp}(V)$, *where q is a power of a prime $p \notin \pi(\mathfrak{X})$, and V is the vector space of dimension $2n$ over $\mathbb{F}_q$ with nondegenerate skew-symmetric form associated with* $\mathrm{PSp}_{2n}(q)$.

(ii) $\pi(\mathfrak{X}) \cap \pi(S) \subseteq \pi(q^2 - 1)$ *and*

– *either* $\pi(\mathfrak{X}) \cap \pi(S) = \{2, 3\}$ *and* $n \in \{5, 7\}$,

– *or* $\pi(\mathfrak{X}) \cap \pi(S) = \{2, 3, 5\}$ *and* $n = 7$.

(iii) *Any $\pi(\mathfrak{X})$-Hall subgroup of the group $\mathrm{PSp}_{2n}(q)$ is contained in the stabilizer M of a decomposition of the space V in the orthogonal sum*

$$V = V_1 \perp \cdots \perp V_n$$

of nondegenerate isometric subspaces of dimension 2. There exists a subgroup $A \trianglelefteq M$ such that $A = L_1 \dots L_n$, where $L_i \cong \mathrm{Sp}(V_i) \cong \mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$, $[L_i, L_j] = 1$, $i, j = 1, \dots, n$, $i \neq j$, and $M/A \cong \mathrm{Sym}_n$.

(iv) $\mathrm{h}_{\mathfrak{X}}(\mathrm{Sym}_n)=1$. *In addition,*
- *if* $\pi(\mathfrak{X})\cap\pi(S)=\{2,3\}$, *then any* $\mathfrak{X}$*-Hall subgroup of the group* Sym_n *is isomorphic to* Sym_4 *for* $n=5$ *and to* $\mathrm{Sym}_3\times\mathrm{Sym}_4$ *for* $n=7$;
- *if* $\pi(\mathfrak{X})\cap\pi(S)=\{2,3,5\}$, *then any* $\mathfrak{X}$*-Hall subgroup of the group* $\mathrm{Sym}_n=\mathrm{Sym}_7$ *is isomorphic to* Sym_6 (*in particular*, $\mathrm{Sym}_m\in\mathfrak{X}$ *for* $m\leqslant 6$).

(v) $\mathrm{h}_{\mathfrak{X}}(\mathrm{Sp}_2(q))=3$. *In addition,*
- *if* $\pi(\mathfrak{X})\cap\pi(S)=\{2,3\}$, *then all* $\mathfrak{X}$*-Hall subgroups in* $\mathrm{Sp}_2(q)\cong\mathrm{SL}_2(q)$ *are solvable*, $\mathrm{Sp}_2(q)$ *contains one class of conjugate* $\mathfrak{X}$*-Hall subgroups isomorphic to the generalized quaternion group*[6] *of order* 48, *and two classes of* $\mathfrak{X}$*-Hall subgroups isomorphic to* $2.\,\mathrm{Sym}_4$;
- *if* $\pi(\mathfrak{X})\cap\pi(S)=\{2,3,5\}$, *then the group* $\mathrm{Sp}_2(q)\cong\mathrm{SL}_2(q)$ *contains a conjugacy class of solvable* $\mathfrak{X}$*-Hall subgroups isomorphic to the generalized quaternion group of order* 120, *and two classes of* $\mathfrak{X}$*-Hall subgroups isomorphic to* $\mathrm{SL}_2(5)\cong 2.\,\mathrm{Alt}_5$.

(vi) *The number of fixed points of any subgroup* $G\leqslant\mathrm{Aut}(S)$ *under its action on the set* $\mathrm{Hall}_{\mathfrak{X}}(S)/S$ *is either* 1 *or* 9.

§ 3. Frattini argument for $\mathfrak{X}$-Hall subgroups

Our main purpose in this section is to prove the following result.

Proposition 2. *Let a group* G *have a normal subgroup* A *such that* $A=KN$ *for some subgroup* N *normal in* G, *where* N *is a direct product of nonabelian simple groups and some* $K\in\mathrm{Hall}_{\mathfrak{X}}(G)$. *Then there exists* $L\in\mathrm{Hall}_{\mathfrak{X}}(A)$ *such that* $G=A\mathrm{N}_G(L)$.

Proof. Let $\pi=\pi(\mathfrak{X})$. Since A contains $H\in\mathrm{Hall}_{\mathfrak{X}}(G)$, the index $|G:A|$ is a π'-number. Since $A\trianglelefteq G$, we have $\mathrm{Hall}_{\mathfrak{X}}(G)=\mathrm{Hall}_{\mathfrak{X}}(A)$, and G/A is a π'-group.

Let $N=S_1\times\dots\times S_n$, where S_i, $i=1,\dots,n$, are nonabelian simple groups. Let us establish some facts on $\mathfrak{X}$-Hall subgroups of S_i, on conjugacy classes of such subgroups, and on the action on these classes of the groups of G-induced automorphisms. We fix $S\in\{S_1,\dots,S_n\}$. Since $K\cap S\in\mathrm{Hall}_{\mathfrak{X}}(S)$, we have $S\in\mathscr{E}_{\mathfrak{X}}$, and hence, by Lemma 8,

$$\mathrm{h}_{\mathfrak{X}}(S)\in\{1,2,3,4,9\}.$$

Let Ω be the set of all fixed points of the group $\mathrm{Aut}_A(S)$ acting on the set $\mathrm{Hall}_{\mathfrak{X}}(S)/S$ of conjugacy classes of $\mathfrak{X}$-Hall subgroups of the group S, that is,

$$\Omega=\{H^S\mid H\in\mathrm{Hall}_{\mathfrak{X}}(S),\ \forall\, a\in\mathrm{N}_A(S)\ \exists\, x\in S\colon H^a=H^x\}.$$

Note that $\Omega\neq\varnothing$, because $(K\cap S)^S\in\Omega$. Indeed, $N\leqslant\mathrm{N}_A(S)$, and hence, $\mathrm{N}_A(S)=\mathrm{N}_{KN}(S)=\mathrm{N}_K(S)N$. In addition, the conjugacy class $(K\cap S)^S$ is invariant under both groups $\mathrm{N}_K(S)$ and N. Hence it is invariant under both $\mathrm{N}_A(S)$ and $\mathrm{Aut}_A(S)$, and, therefore, lies in Ω.

Since $A\trianglelefteq G$, we have $\mathrm{N}_A(S)\trianglelefteq\mathrm{N}_G(S)$. Hence $\mathrm{Aut}_A(S)\trianglelefteq\mathrm{Aut}_G(S)$ and, therefore, the group $\mathrm{Aut}_G(S)$ acts on Ω. We assert that

[6] A generalized quaternion group is understood according to the definition in [19], Ch. II, § 9, pp. 258–259.

1°) *the length of some orbit of the group* $\mathrm{Aut}_G(S)$ *on* Ω *is a* π*-number.*

Indeed, $|\Omega| \leqslant \mathrm{h}_{\mathfrak{X}}(S)$, and the length of any orbit in $\mathrm{Aut}_G(S)$ on Ω is at most $|\Omega|$. If $2 \notin \pi$ or $3 \notin \pi$, then from Lemma 8 it follows that the length of any orbit of the group $\mathrm{Aut}_G(S)$ on Ω is a π-number. So, we can assume that $2, 3 \in \pi$. Now if $\mathrm{h}_{\mathfrak{X}}(S) \leqslant 4$, then the length of any orbit of the group $\mathrm{Aut}_G(S)$ on Ω is again a π-number. So, we can assume that $\mathrm{h}_{\mathfrak{X}}(S) = 9$. From Lemma 9 it follows that $|\Omega| \in \{1, 9\}$. The case $|\Omega| = 1$ is clear. The only non-$\{2, 3\}$-numbers majorized by 9 are 5 and 7, and so, as is easily checked, any partition of 9 into a sum of natural numbers involves a $\{2, 3\}$-number and, therefore, a π-number. Hence the length of one of the orbits into which Ω splits relative to the action of$\mathrm{Aut}_G(S)$ is a π-number.

Assertion 1°) can be refined as follows:

2°) *if the length of an orbit of* $\mathrm{Aut}_G(S)$ *on* Ω *is a* π*-number, then this length is* 1.

The hypotheses of the theorem imply that G/A is a π'-group. So, both the group $\mathrm{N}_G(S)A/A \cong \mathrm{N}_G(S)/\mathrm{N}_A(S)$ and its homomorphic image

$$\begin{aligned}\mathrm{N}_G(S)/\mathrm{N}_A(S)\mathrm{C}_G(S) &\cong (\mathrm{N}_G(S)/\mathrm{C}_G(S))/(\mathrm{N}_A(S)\mathrm{C}_G(S)/\mathrm{C}_G(S))\\ &\cong \mathrm{Aut}_G(S)/\,\mathrm{Aut}_A(S)\end{aligned}$$

are also π'-groups. By definition of the set Ω, $\mathrm{Aut}_A(S)$ stabilizes any element from Ω. Hence the length of any orbit on Ω of the group $\mathrm{Aut}_G(S)$ divides a π'-number $|\mathrm{Aut}_G(S) : \mathrm{Aut}_A(S)|$ and, therefore, is itself a π'-number. If a number is simultaneously a π- and a π'-number, then it is equal to 1.

From 1°) and 2°) we conclude that

3°) *there exist* $H \in \mathrm{Hall}_{\mathfrak{X}}(S)$ *such that, for any* $\gamma \in \mathrm{Aut}_G(S)$, *the subgroup* H^γ *is conjugate in* S *to* H.

We now assert the following:

4°) *any minimal normal subgroup* M *of the group* G *such that* $M \leqslant N$ *contains a subgroup* $V_M \in \mathrm{Hall}_{\mathfrak{X}}(M)$ *such that* $V_M^M = V_M^G$.

It can be assumed that $M = \langle S^G \rangle$. Assertion 4°) follows from Lemma 4.

We also assert that

5°) *there exists* $V \in \mathrm{Hall}_{\mathfrak{X}}(N)$ *such that* $V^N = V^G$.

By Λ we denote the set of all minimal normal subgroups of the group G lying in N. By the assumption,

$$N = \prod_{M \in \Lambda} M,$$

where the product is direct. For each $M \in \Lambda$, we choose, according to 4°), a subgroup $V_M \in \mathrm{Hall}_{\mathfrak{X}}(M)$ so that $V_M^M = V_M^G$. Let

$$V = \langle V_M \mid M \in \Lambda \rangle.$$

Then $V \in \mathrm{Hall}_{\mathfrak{X}}(N)$. Consider any $g \in G$. Any $M \in \Lambda$ contains an element x_M such that $V_M^g = V_M^{x_M}$. We set

$$x = \prod_{M \in \Lambda} x_M.$$

It is clear that $x \in N$ and $V_M^x = V_M^{x_M} = V_M^g$ for any $M \in \Lambda$. Therefore,

$$V^g = \langle V_M^g \mid M \in \Lambda \rangle = \langle V_M^x \mid M \in \Lambda \rangle = V^x.$$

This proves assertion 5°).

6°) *The groups* $\mathrm{N}_A(V)$ *and* $\mathrm{N}_G(V)$ *are* $\mathfrak{X}$*-separable.*

Consider the normal series

$$\mathrm{N}_G(V) \trianglerighteq \mathrm{N}_A(V) \trianglerighteq \mathrm{N}_N(V) \trianglerighteq V \trianglerighteq 1$$

and the sections of this series. The section $\mathrm{N}_G(V)/\mathrm{N}_A(V)$ is isomorphic to the subgroup $\mathrm{N}_G(V)A/A$ in the $\mathfrak{X}'$-group G/A and, therefore, is also an $\mathfrak{X}'$-group. Similarly

$$\mathrm{N}_A(V)/\mathrm{N}_N(V) \cong \mathrm{N}_A(V)N/N \leqslant A/N = KN/N \cong K/(K \cap N),$$

which gives $\mathrm{N}_A(V)/\mathrm{N}_N(V) \in \mathfrak{X}$. Since $V \in \mathrm{Hall}_{\mathfrak{X}}(N)$, we have $\mathrm{N}_N(V)/V \in \mathfrak{X}'$. Finally, $V \in \mathfrak{X}$. This proves 6°).

Now the required proposition follows from 5°) and 6°). Using 5°), we see that $V^A = V^N$. By Lemma 2, there exists $L \in \mathrm{Hall}_{\mathfrak{X}}(A)$ such that $V = L \cap N$. Let us show that L is as claimed in the proposition. It suffices to prove the inclusion $G \leqslant A\mathrm{N}_G(L)$. It is clear that $L \leqslant \mathrm{N}_A(V)$, that is, L is an $\mathfrak{X}$-Hall subgroup of the $\mathfrak{X}$-separable normal subgroup $\mathrm{N}_A(V)$ of the group $\mathrm{N}_G(V)$. From conjugacy of the $\mathfrak{X}$-Hall subgroups in $\mathfrak{X}$-separable groups, we have

$$L^{\mathrm{N}_G(V)} = L^{\mathrm{N}_A(V)}, \quad \text{which implies} \quad \mathrm{N}_G(V) \leqslant \mathrm{N}_A(V)\mathrm{N}_G(L).$$

Now an appeal to 5°) shows that

$$G = N\mathrm{N}_G(V) \leqslant N\mathrm{N}_A(V)\mathrm{N}_G(L) \leqslant A\mathrm{N}_G(L),$$

which completes the proof of Proposition 2.

It seems that by using [24], Theorem 3.1 (see also [25], Theorem 2), Proposition 2 might be strengthened to the hollowing hypothetical result: *if* $G \in \mathscr{E}_{\mathfrak{X}}$ *and* $A \trianglelefteq G$, *then* $G = A\mathrm{N}_G(H)$ *for some* $H \in \mathrm{Hall}_{\mathfrak{X}}(A)$. If $\mathfrak{X}$ is the class of π-groups, this result is known (see [12], Corollary 3.7).

§ 4. On simple groups with nine conjugacy classes of $\mathfrak{X}$-Hall subgroups

Proposition 3. *Let* $\mathfrak{X}$ *be a complete class of finite groups*, S *be a nonabelian simple group and* $\mathrm{h}_{\mathfrak{X}}(S) = 9$. *Then* $S \notin \mathscr{M}_{\mathfrak{X}}$.

Proof. Assume that $S \in \mathscr{M}_{\mathfrak{X}}$. Let $\pi = \pi(\mathfrak{X})$. In view of Lemma 9, we can assume that

$$S = \mathrm{PSp}_{2n}(q) \cong \mathrm{PSp}(V) \quad \text{and} \quad \pi(\mathfrak{X}) \cap \pi(S) \subseteq \pi(q^2-1) \subseteq \pi(\mathrm{SL}_2(q)).$$

Any $\mathfrak{X}$-Hall subgroup of the group $\mathrm{PSp}_{2n}(q)$ is contained in the stabilizer M of a decomposition of the associated space V in the orthogonal sum

$$V = V_1 \perp \cdots \perp V_n$$

of nondegenerate isometric subspaces of dimension 2. There exists a subgroup $A \trianglelefteq M$ such that $A = L_1 \dots L_n$, where $L_i \cong \mathrm{Sp}(V_i) \cong \mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$, $[L_i, L_j] = 1$, $i, j = 1, \dots, n$, $i \neq j$, and $M/A \cong \mathrm{Sym}_n$.

One of the following two cases holds.

(1) $\pi(\mathfrak{X}) \cap \pi(S) = \pi(\mathfrak{X}) \cap \pi(\mathrm{SL}_2(q)) = \{2, 3\}$ and $n \in \{5, 7\}$. In addition, the $\mathfrak{X}$-Hall subgroups of any group $L_i \cong \mathrm{SL}_2(q)$ are, precisely, the generalized quaternion groups of order 48 and groups of the form $2.\mathrm{Sym}_4$.

(2) $\pi(\mathfrak{X}) \cap \pi(S) = \pi(\mathfrak{X}) \cap \pi(\mathrm{SL}_2(q)) = \{2, 3, 5\}$ and $n = 7$. In addition, the $\mathfrak{X}$-Hall subgroups in any $L_i \cong \mathrm{SL}_2(q)$ are, precisely, the generalized quaternion groups of order 120 and groups of the form $2.\mathrm{Alt}_5$; any $\mathfrak{X}$-Hall subgroup in $M/A \cong \mathrm{Sym}_7$ is isomorphic to Sym_6.

In each of these cases, we choose in S a subgroup U as follows.

Consider case (1). The group S contains a subgroup of the form

$$\mathrm{Sp}_6(q) \circ \mathrm{Sp}_{2(n-3)}(q)$$

which stabilizes in S a nondegenerate subspace of dimension 6 and its orthogonal complement, and hence, contains a subgroup isomorphic to $\mathrm{Sp}_6(q)$. For any $\varepsilon \in \{+, -\}$, this subgroup contains a subgroup[7] $\mathrm{GL}_3^{\varepsilon}(q).2$ (see [18], Table 8.28); here ε can be chosen so that the number $q - \varepsilon 1$ would be divisible by 3. With this choice of ε, in view of [18], Tables 8.3, 8.5, we can choose a $\{2, 3\}$-subgroup

$$U := 3_+^{1+2} : Q_8$$

in the subgroup $\mathrm{SL}_3^{\varepsilon}(q) \leqslant \mathrm{GL}_3^{\varepsilon}(q).2$. By solvability, $U \in \mathfrak{X}$. Since $S \in \mathscr{M}_{\mathfrak{X}}$, we have $U \leqslant H$ for some $H \in \mathrm{Hall}_{\mathfrak{X}}(S)$. Proceeding as above, we choose a subgroup M and a normal subgroup A in M such that $H \leqslant M$. Consider the canonical epimorphism

$$\overline{} \colon M \to M/A.$$

We have $\overline{U} \leqslant \overline{H} \leqslant \overline{M} \cong \mathrm{Sym}_n$, where $n \in \{5, 7\}$. On the other hand, $\overline{H} \cong H/(H \cap A)$ and $\overline{U} \cong U(H \cap A)/(H \cap A)$. We choose in $H \cap A$ the characteristic subgroups B, C, and D defined by

$$B := \mathrm{O}_2(H \cap A), \quad C/B := \mathrm{O}_3((H \cap A)/B) \quad \text{and} \quad D/C := \mathrm{O}_2((H \cap A)/C).$$

By the choice, $B \leqslant C \leqslant D$. From Lemma 2 it follows that the subgroup $H \cap A$ is generated by pairwise permutational $\mathfrak{X}$-Hall subgroups of factors L_i, each of which is either a generalized quaternion $\{2, 3\}$-group, or is isomorphic to $2.\mathrm{Sym}_4$. Now it is clear that $D = H \cap A$, and, therefore, $\overline{U} \cong U/(U \cap D)$. Since $\mathrm{O}_2(U) = 1$, we have

$$U \cap B = 1 \quad \text{and} \quad U \cong UB/B.$$

Since, in each factor, the Sylow 3-subgroups forming the group $H \cap A$ are cyclic groups of order 3, any Sylow 3-subgroup of the group $H \cap A$ which is isomorphic to C/B is abelian, and its section

$$(UB/B) \cap (C/B) = (U \cap C)B/B \cong U \cap C$$

[7] Here, we follow the standard approach adopted for classical finite groups (see [18], for example) by putting $\mathrm{GL}_m^+(q) = \mathrm{GL}_m(q)$, $\mathrm{SL}_m^+(q) = \mathrm{SL}_m(q)$, $\mathrm{GL}_m^-(q) = \mathrm{GU}_m(q)$ and $\mathrm{SL}_n^-(q) = \mathrm{SU}_n(q)$.

is a normal abelian 3-subgroup of the group $UB/B \cong U$. It follows that it is contained in $\mathrm{Z}(\mathrm{O}_3(UB/B))$, inasmuch as

$$UB/B \cong 3_+^{1+2} : Q_8,$$

and Q_8 acts irreducibly on the quotient group of the group 3_+^{1+2} to its centre. Therefore,

$$\text{either} \quad UC/C \cong U/(U \cap C) \cong 3_+^{1+2} : Q_8, \quad \text{or} \quad UC/C \cong 3^2 : Q_8.$$

Now, since $\mathrm{O}_2(3_+^{1+2} : Q_8) = 1$ and $\mathrm{O}_2(3^2 : Q_8) = 1$, we find that

$$(UC/C) \cap D/C = 1 \quad \text{and} \quad \overline{U} = UD/D \cong UC/C.$$

But $\overline{U}$ (and, therefore, its subgroup Q_8) is isomorphic to a subgroup of the group Sym_n for $n \in \{5, 7\}$. At the same time, it is quite clear that the group Q_8 has not faithful permutation representations of degree < 8. A contradiction.

Consider case (2). Proceeding with the group $S = \mathrm{PSp}_{14}(q)$ as in case (1), we find a subgroup isomorphic to $\mathrm{Sp}_{10}(q)$. Since 5 divides $q^2 - 1$, we choose $\varepsilon \in \{+, -\}$ so that 5 would divide $q - \varepsilon 1$. The group $\mathrm{Sp}_{10}(q)$, and hence, the group S, contains a subgroup $\mathrm{GL}_5^{\varepsilon}(q).2$ (see [18], Table 8.64), which, in turn, contains $\mathrm{SL}_5^{\varepsilon}(q)$ as a subgroup. Further, $\mathrm{SL}_5^{\varepsilon}(q)$ contains a subgroup

$$U := 5_+^{1+2}.\,\mathrm{Sp}_2(5),$$

see [18], Tables 8.18 and 8.20. Moreover, since $\mathrm{Sp}_2(5) \cong 2.\,\mathrm{Alt}_5$, we have $U \in \mathfrak{X}$. Next, since $S \in \mathscr{M}_{\mathfrak{X}}$, we see that $U \leqslant H$ for some $H \in \mathrm{Hall}_{\mathfrak{X}}(S)$. Proceeding as above, we choose a subgroup M and in it a normal subgroup A so as to have $H \leqslant M$. Let

$$^{-}\colon M \to M/A$$

be the canonical epimorphism. Then $\overline{U} \leqslant \overline{H} \leqslant \overline{M} \cong \mathrm{Sym}_7$. Therefore, $|\overline{U}|_5 \leqslant 5$. From the structure of the group U any homomorphism image of the group U whose order is not divisible by 5^2 is an image of the group $\mathrm{Sp}_2(5) \cong \mathrm{SL}_2(5)$. Hence the extra special subgroup 5_+^{1+2} of the group U should lie in the kernel of the homomorphism $^{-}$, and hence, in $U \cap A$. But the Sylow 5-subgroups of the group A are abelian (in each factor L_i the order of the Sylow 5-subgroup is 5). This contradiction proves Proposition 3.

§ 5. Proof of Theorem 1 and its corollaries

Proof of Theorem 1. Assume on the contrary that there exists a group G with the following properties:

(a) G has a normal subgroup N such that $\mathrm{k}_{\mathfrak{X}}(N) > 1$, but the reduction $\mathfrak{X}$-theorem holds for the pair (G, N), that is, $\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{k}_{\mathfrak{X}}(\overline{G})$, where bar

$$^{-}\colon G \to G/N$$

denotes the canonical epimorphism;

(b) the order of G is smallest among the groups with property (a).

Recall (see § 1.2) that the reduction $\mathfrak{X}$-theorem for the pair (G, N) implies the following properties:

1°) *if* $K \in \mathrm{m}_{\mathfrak{X}}(G)$, *then* $\overline{K} \in \mathrm{m}_{\mathfrak{X}}(\overline{G})$;

2°) *if for* $K, L \in \mathrm{m}_{\mathfrak{X}}(G)$, *the subgroups* $\overline{K}$ *and* $\overline{L}$ *are conjugate in* $\overline{G}$ (*for example, are equal*), *then* K *and* L *are conjugate in* G.

Note that if $M \neq 1$ is a normal subgroup of G, $M \leqslant N$, then, since G/N is a homomorphic image of the group G/M, we have

$$\mathrm{k}_{\mathfrak{X}}(G) = \mathrm{k}_{\mathfrak{X}}(G/N) \leqslant \mathrm{k}_{\mathfrak{X}}(G/M) \leqslant \mathrm{k}_{\mathfrak{X}}(G).$$

As a result, we get the reduction $\mathfrak{X}$-theorem for the pairs $(G/M, N/M)$ and (G, M). By (b) and since $|G/M| < |G|$, we have $\mathrm{k}_{\mathfrak{X}}(N/M) = 1$. Hence if $\mathrm{k}_{\mathfrak{X}}(M) = 1$, then by Lemma 3

$$\mathrm{k}_{\mathfrak{X}}(N) = \mathrm{k}_{\mathfrak{X}}(N/M) = 1.$$

Therefore, it can be assumed that

3°) N *is a minimal normal subgroup of the group* G, *and* N *is nonabelian*, because $\mathrm{k}_{\mathfrak{X}}(N) > 1$. Therefore, according to [19], Ch. 2, Corollary 3 to Theorem 4.14,

$$N = S_1 \times \cdots \times S_n$$

for some nonabelian simple subgroups $S_1, \ldots, S_n$ conjugate in G. Let S be one of the S_i's.

We will obtain a contradiction by examining the action of the group $\mathrm{Aut}_G(S)$ on the set

$$\Delta := \mathrm{Hall}_{\mathfrak{X}}(S)/S$$

of the conjugacy classes of $\mathfrak{X}$-Hall subgroups of the group S. By Lemma 8,

4°) $|\Delta| = \mathrm{h}_{\mathfrak{X}}(S) \in \{0, 1, 2, 3, 4, 9\}$.

Let us exclude all six possibilities. We will first verify that

5°) $\mathrm{h}_{\mathfrak{X}}(S) \neq 0$.

To this end, we will show that

6°) *if* $K \in \mathrm{m}_{\mathfrak{X}}(G)$, *then* $K \cap N \in \mathrm{Hall}_{\mathfrak{X}}(N)$ *and* $K \in \mathrm{Hall}_{\mathfrak{X}}(KN)$.

As a result, $\mathrm{Hall}_{\mathfrak{X}}(N) \neq \varnothing$ and $\mathrm{h}_{\mathfrak{X}}(S) \neq 0$, because

$$\varnothing \neq \{H \cap S \mid H \in \mathrm{Hall}_{\mathfrak{X}}(N)\} \subseteq \mathrm{Hall}_{\mathfrak{X}}(S).$$

In addition, from the inclusion $K \cap N \in \mathrm{Hall}_{\mathfrak{X}}(N)$ it will also follow that $K \in \mathrm{Hall}_{\mathfrak{X}}(KN)$, inasmuch as

$$|KN : K| = \frac{|K||N|}{|K \cap N|} : |K| = |N : (K \cap N)|.$$

We choose an arbitrary $K \in \mathrm{m}_{\mathfrak{X}}(G)$, $p \in \pi(\mathfrak{X})$ and $P \in \mathrm{Syl}_p(N)$. We have $P \in \mathfrak{X}$. It suffices to prove that P is conjugate to a subgroup from K. We set

$$A := KN.$$

From Frattini's argument (see [15], Ch. A, (6.3)) it follows that $A = \mathrm{N}_A(P)N$. Hence

$$\overline{K} = \overline{A} = \overline{\mathrm{N}_A(P)}$$

and now, by Lemma 1, we have $\overline{\mathrm{N}_A(P)} = \overline{U}$ for some $U \in \mathrm{m}_{\mathfrak{X}}(\mathrm{N}_A(P))$. Since U normalizes the $\mathfrak{X}$-subgroup P, we have $P \leqslant U$. We embed U into a maximal $\mathfrak{X}$-subgroup L of the group G. By 1°), we have $\overline{L} \in \mathrm{m}_{\mathfrak{X}}(\overline{G})$. In addition,

$$\overline{K} = \overline{\mathrm{N}_A(P)} = \overline{U} \leqslant \overline{L}.$$

Similarly by 1°) and since $K \in \mathrm{m}_{\mathfrak{X}}(G)$, we have $\overline{K} \in \mathrm{m}_{\mathfrak{X}}(\overline{G})$. Now we have $\overline{K} = \overline{L}$, and hence, using 2°), we conclude that $L^g = K$ for some $g \in G$. Hence

$$P^g \leqslant U^g \leqslant L^g = K,$$

as claimed.

We note some other corollaries to 6°). We assert that

7°) *any $\mathfrak{X}$-subgroup in G normalizes some member of* $\mathrm{Hall}_{\mathfrak{X}}(N)$; *in particular*

8°) $N \in \mathscr{M}_{\mathfrak{X}}$ *and* $S \in \mathscr{M}_{\mathfrak{X}}$;

9°) *any $\mathfrak{X}$-subgroup in* $\mathrm{Aut}_G(S)$ *stabilizes some element from* Δ.

Let us verify 7°) and 8°). If U is an $\mathfrak{X}$-subgroup of the group G, then $U \leqslant K$ for some $K \in \mathrm{m}_{\mathfrak{X}}(G)$ and U normalizes $K \cap N \in \mathrm{Hall}_{\mathfrak{X}}(N)$ in view of 6°). This proves 7°). If here we take $U \in \mathrm{m}_{\mathfrak{X}}(N)$, then $U(K \cap N)$ is an $\mathfrak{X}$-subgroup in N containing $U \in \mathrm{m}_{\mathfrak{X}}(N)$ and $K \cap N \in \mathrm{Hall}_{\mathfrak{X}}(N) \subseteq \mathrm{m}_{\mathfrak{X}}(N)$, and hence

$$U = U(K \cap N) = K \cap N \in \mathrm{Hall}_{\mathfrak{X}}(N).$$

This shows that $\mathrm{m}_{\mathfrak{X}}(N) = \mathrm{Hall}_{\mathfrak{X}}(N)$. Therefore, $N \in \mathscr{M}_{\mathfrak{X}}$. From Lemma 2 it follows that any normal subgroup of an $\mathscr{M}_{\mathfrak{X}}$-group is an $\mathscr{M}_{\mathfrak{X}}$-group. Hence $S \in \mathscr{M}_{\mathfrak{X}}$, as claimed in 8°).

Let us verify 9°). We denote by

$$\alpha \colon \mathrm{N}_G(S) \to \mathrm{Aut}_G(S)$$

the epimorphism which associates with each $g \in \mathrm{N}_G(S)$ the automorphism of the group S acting by the rule $x \mapsto x^g$ for all $x \in S$. By Lemma 1, an arbitrary $\mathfrak{X}$-subgroup T of the group $\mathrm{Aut}_G(S)$ has the form U^α, where U is some $\mathfrak{X}$-subgroup of the group $\mathrm{N}_G(S)$. Let $t \in T$, and let that $t = g^\alpha$ for $g \in U$. Then $x^t = x^{g^\alpha} = x^g$ for all $x \in S$. By 7°), the subgroup U normalizes some $H \in \mathrm{Hall}_{\mathfrak{X}}(N)$. Since U also normalizes S, we have

$$(H \cap S)^t = (H \cap S)^g = H \cap S.$$

Therefore, T leaves invariant the conjugacy class of $\mathfrak{X}$-Hall subgroups of the group S that contains $H \cap S \in \mathrm{Hall}_{\mathfrak{X}}(S)$.

10°) $\mathrm{h}_{\mathfrak{X}}(S) \neq 1$.

Indeed, if $\mathrm{h}_{\mathfrak{X}}(S) = 1$, then $S \in \mathscr{C}_{\mathfrak{X}}$, and hence, by 8°), we have

$$S \in \mathscr{C}_{\mathfrak{X}} \cap \mathscr{M}_{\mathfrak{X}} = \mathscr{D}_{\mathfrak{X}}.$$

But then $\mathrm{k}_{\mathfrak{X}}(S_1) = \cdots = \mathrm{k}_{\mathfrak{X}}(S_n) = 1$ and $\mathrm{k}_{\mathfrak{X}}(N) = 1$ by Lemma 3, in contrast to (a).

11°) $\mathrm{h}_{\mathfrak{X}}(S) \neq 9$.

We have $S \in \mathscr{M}_{\mathfrak{X}}$. But if $\mathrm{h}_{\mathfrak{X}}(S) = 9$, then $S \notin \mathscr{M}_{\mathfrak{X}}$ by Proposition 3.

12°) $\mathrm{N}_G(KN) \in \mathscr{C}_{\mathfrak{X}}$ *and* $KN \in \mathscr{C}_{\mathfrak{X}}$ *for any* $K \in \mathrm{m}_{\mathfrak{X}}(G)$.

Let $K \in \mathrm{m}_{\mathfrak{X}}(G)$. We set

$$A := KN \quad \text{and} \quad B := \mathrm{N}_G(A).$$

Since $\overline{A} = \overline{K} \in \mathrm{m}_{\mathfrak{X}}(\overline{G})$, we conclude that $B/A \cong \mathrm{N}_{\overline{G}}(\overline{K})/\overline{K}$ is a π'-group. Therefore,

$$\mathrm{Hall}_{\mathfrak{X}}(A) = \mathrm{Hall}_{\mathfrak{X}}(B).$$

According to 6°) we have $K \in \mathrm{Hall}_{\mathfrak{X}}(A)$. In particular, $\mathrm{Hall}_{\mathfrak{X}}(A) \neq \varnothing$. Therefore,

$$A, B \in \mathscr{E}_{\mathfrak{X}} \quad \text{and} \quad \mathrm{h}_{\mathfrak{X}}(A) \geqslant \mathrm{h}_{\mathfrak{X}}(B) \geqslant 1.$$

Given an arbitrary $L \in \mathrm{Hall}_{\mathfrak{X}}(A)$, we will show that L and K are conjugate in B. First, it is clear that $A = KN = LN$, and hence, $\overline{K} = \overline{L}$. Therefore,

$$\overline{L} \in \mathrm{m}_{\mathfrak{X}}(\overline{G}) \quad \text{and} \quad L \cap N \in \mathrm{Hall}_{\mathfrak{X}}(N) \subseteq \mathrm{m}_{\mathfrak{X}}(N).$$

Hence $L \in \mathrm{m}_{\mathfrak{X}}(G)$. According to 2°), the equality $\overline{K} = \overline{L}$ implies the conjugacy of the subgroups K and L in G and, therefore, in $B = \mathrm{N}_G(KN) = \mathrm{N}_G(LN)$. So, we have shown that the group B acts transitively by conjugations on the nonempty set $\mathrm{Hall}_{\mathfrak{X}}(A) = \mathrm{Hall}_{\mathfrak{X}}(B)$. Therefore, $\mathrm{N}_G(KN) = B \in \mathscr{C}_{\mathfrak{X}}$.

From Proposition 2 it follows that *there exists* an $L \in \mathrm{Hall}_{\mathfrak{X}}(A)$ satisfying $B = \mathrm{N}_B(L)A$. Next, since $\mathrm{Hall}_{\mathfrak{X}}(A) = \mathrm{Hall}_{\mathfrak{X}}(B)$ and $B \in \mathscr{C}_{\mathfrak{X}}$, this equality holds *for any* $L \in \mathrm{Hall}_{\mathfrak{X}}(A)$. From $B \in \mathscr{C}_{\mathfrak{X}}$ it also follows that, for any $L \in \mathrm{Hall}_{\mathfrak{X}}(A)$, there exist an element $b \in B$ such that $K = L^b$. In addition, we have $b = ga$ for some $g \in \mathrm{N}_B(L)$ and $a \in A$. Therefore,

$$K = L^b = L^{ga} = L^a, \quad \text{where} \quad a \in A,$$

and which shows that $KN = A \in \mathscr{C}_{\mathfrak{X}}$, as claimed.

From 12°) we can establish the following fact, which will be crucial in the proof of the remaining cases:

13°) *if* $K \in \mathrm{m}_{\mathfrak{X}}(G)$, *then* $\mathrm{Aut}_K(S)$ *stabilizes precisely one element from* Δ.

Let $K \in \mathrm{m}_{\mathfrak{X}}(G)$. That Δ has a fixed point for $\mathrm{Aut}_K(S)$ follows from 9°). Let $H \in \mathrm{Hall}_{\mathfrak{X}}(S)$. We will prove 13°) if we establish that the invariancy of the class $H^S \in \mathrm{Hall}_{\mathfrak{X}}(S)/S$ with respect to $\mathrm{Aut}_K(S)$ implies that $H \in (K \cap S)^S$. Let, as before, $A = KN$. By Lemma 5, we have

$$H^{\mathrm{Aut}_A(S)} = (H^S)^{\mathrm{Aut}_K(S)} = H^S.$$

Now from Lemma 4 it follows that, for

$$M := \langle S^A \rangle = \langle S^K \rangle,$$

there exist $L \in \mathrm{Hall}_{\mathfrak{X}}(KM)$ such that $H = L \cap S$. In addition, $|L| = |K|$, and, therefore, $L \in \mathrm{Hall}_{\mathfrak{X}}(A)$. But $A = KN \in \mathscr{C}_{\mathfrak{X}}$ by 12°), and hence, there exist $u \in K$ and $v \in N$ such that $L = K^{uv} = K^v$. It is clear that if $w \in S$ is the projection of v to S (recall that, S is one of the direct factors $S_1, \ldots, S_n$ whose product makes up the group N), then

$$H = L \cap S = K^v \cap S = (K \cap S)^v = (K \cap S)^w \in (K \cap S)^S.$$

This proves 13°).

From 4°), 5°), 10°) and 11°) it follows that $\mathrm{h}_{\mathfrak{X}}(S) \in \{2, 3, 4\}$. We claim that

14°) $\mathrm{h}_{\mathfrak{X}}(S) \neq 2$.

Indeed, we fix some $K \in \mathrm{m}_{\mathfrak{X}}(G)$. According to 13°), the group $\mathrm{Aut}_K(S)$ stabilizes precisely one element from Δ. But for $\mathrm{h}_{\mathfrak{X}}(S) = 2$ the group $\mathrm{Aut}_K(S)$ also stabilizes the remaining element. A contradiction.

By the above and in view of Lemma 8, it can be assumed that

15°) $\mathrm{h}_{\mathfrak{X}}(S) \in \{3, 4\}$ and $2, 3 \in \pi(\mathfrak{X})$.

Now by Burnside's theorem [15], § I.2, and since each solvable $\pi(\mathfrak{X})$-group is an $\mathfrak{X}$-group, we have

16°) *each* $\{2, 3\}$*-group is an* $\mathfrak{X}$*-group.*

The theorem will be proved once we obtain a contradiction to 9°) by showing that

17°) *some* $\mathfrak{X}$*-subgroup in* $\mathrm{Aut}_G(S)$ *acts transitively on* Δ.

The action of the group $\mathrm{Aut}_G(S)$ on the set Δ induces the homomorphism

$$^*\colon \mathrm{Aut}_G(S) \to \mathrm{Sym}(\Delta), \quad \text{where} \quad \mathrm{Sym}(\Delta) \cong \begin{cases} \mathrm{Sym}_3 & \text{if } \mathrm{h}_{\mathfrak{X}}(S) = 3, \\ \mathrm{Sym}_4 & \text{if } \mathrm{h}_{\mathfrak{X}}(S) = 4. \end{cases}$$

Consider an arbitrary $H \in \mathrm{Hall}_{\mathfrak{X}}(S)$ and consider the class $H^S = \{H^x \mid x \in S\} \in \Delta$. Let $H \leqslant K$ for some $K \in \mathrm{m}_{\mathfrak{X}}(G)$. Since the subgroup $H = K \cap S$ is invariant under $\mathrm{N}_K(S)$, the class H^S is stabilized by the group $\mathrm{Aut}_K(S)$, and according to 11°), this group acts without fixed points on the set

$$\Gamma := \Delta \setminus \{H^S\}$$

of the remaining two or three classes. Since

$$|\Gamma| = \mathrm{h}_{\mathfrak{X}}(S) - 1 = \begin{cases} 2 & \text{if } \mathrm{h}_{\mathfrak{X}}(S) = 3, \\ 3 & \text{if } \mathrm{h}_{\mathfrak{X}}(S) = 4, \end{cases}$$

it follows that the action of $\mathrm{Aut}_K(S)$ on Γ should be transitive. But then so is the action of the stabilizer in $\mathrm{Aut}_G(S)$ of the point H^S on Γ, because this stabilizer contains the subgroup $\mathrm{Aut}_K(S)$. Therefore, $\mathrm{Aut}_G(S)^*$ is a transitive (and even 2-transitive) subgroup in $\mathrm{Sym}(\Delta)$, that is,

$$\mathrm{Aut}_G(S)^* \cong \begin{cases} \mathrm{Sym}_3 & \text{if } \mathrm{h}_{\mathfrak{X}}(S) = 3, \\ \mathrm{Alt}_4 \text{ or } \mathrm{Sym}_4 & \text{if } \mathrm{h}_{\mathfrak{X}}(S) = 4. \end{cases}$$

In any case, $\mathrm{Aut}_G(S)^*$ is a $\{2, 3\}$-group and, therefore, is an $\mathfrak{X}$-group. By Lemma 1, there exists an $\mathfrak{X}$-subgroup U of the group $\mathrm{Aut}_G(S)$ such that $U^* = \mathrm{Aut}_G(S)^*$. This subgroup U acts transitively on Δ, in contrast to 9°).

This completes the proof of Theorem 1.

For proofs of Corollaries 1–3, see §§ 1.2, 1.3.

Proof of Corollary 4. It suffices to show that (i) $\Rightarrow$ (ii). Let $H \in \mathrm{m}_{\mathfrak{X}}(G)$ and $H \leqslant K \leqslant G$. Let a subgroup $N \trianglelefteq G$ be such that $\mathrm{k}_{\mathfrak{X}}(G/N) = \mathrm{k}_{\mathfrak{X}}(G)$. Then $N \in \mathscr{D}_{\mathfrak{X}}$ by Theorem 1. We claim that $\mathrm{k}_{\mathfrak{X}}(K/(K \cap N)) = \mathrm{k}_{\mathfrak{X}}(K)$.

Assume that N is minimal in G as a normal subgroup. Then N is a direct product of its simple subgroups conjugate in G. From Theorem 2 and Lemma 2.28 in [4] we conclude that either $N \in \mathfrak{X}$, or, for $\pi = \pi(\mathfrak{X})$, any π-Hall subgroup of the group N is solvable (in particular, lies in $\mathfrak{X}$) and $N \in \mathscr{D}_\pi$. In the first case, the required result is clear. In the second case, from Lemma 2 it follows that $H \cap N \in \mathrm{Hall}_{\mathfrak{X}}(N) \subseteq \mathrm{Hall}_\pi(N)$ and $H \cap N \leqslant K \cap N \leqslant N$. According to Theorem 1.4, [16], $K \cap N \in \mathscr{D}_\pi$. Since any π-Hall subgroup from $K \cap N$ lies in $\mathfrak{X}$, we have $K \cap N \in \mathscr{D}_{\mathfrak{X}}$. As a result, $\mathrm{k}_{\mathfrak{X}}(K/(K \cap N)) = \mathrm{k}_{\mathfrak{X}}(K)$ by Theorem 2.

The general case can be derived from that considered above by induction on $|N|$ after passing to the quotient group relative to the minimal normal subgroup of the group G contained in N, and then applying Theorem 2. This proves the corollary.

Proof of Corollary 5. By definition of the relation $\underset{\mathfrak{X}}{\equiv}$, $G_1 \underset{\mathfrak{X}}{\equiv} G_2$ if and only if the complete $\mathfrak{X}$-reductions of the groups G_1 and G_2 are isomorphic. Now all the assertions in Corollary 5 are clear. This proves the corollary.

Bibliography

[1] H. Wielandt, “On the structure of composite groups”, *Proceedings of the international conference on the theory of groups* (Austral. Nat. Univ., Canberra 1965), Gordon and Breach Science Publishers Inc., New York 1967, pp. 379–388.

[2] H. Wielandt, “Zusammengesetzte Gruppen endlicher Ordnung”, Lecture notes, Math. Inst. Univ. Tübingen, 1963/64, *Mathematische Werke / Mathematical works*, vol. 1: *Group theory*, Walter de Gruyter & Co., Berlin 1994, pp. 607–655.

[3] H. Wielandt, “Zusammengesetzte Gruppen: Hölders Programm heute”, *The Santa Cruz conference on finite groups* (Univ. California, Santa Cruz, CA 1979), Proc. Sympos. Pure Math., vol. 37, Amer. Math. Soc., Providence, RI 1980, pp. 161–173.

[4] Wenbin Guo, D. O. Revin, and E. P. Vdovin, “The reduction theorem for relatively maximal subgroups”, *Bull. Math. Sci.* **12**:1 (2022), 2150001.

[5] M. L. Sylow, “Théorèmes sur les groupes de substitutions”, *Math. Ann.* **5**:4 (1872), 584–594.

[6] P. Hall, “A note on soluble groups”, *J. London Math. Soc.* **3**:2 (1928), 98–105.

[7] S. A. Chunikhin, “On Π-separable groups”, *Dokl. Akad. Nauk SSSR* **59**:3 (1948), 443–445.

[8] S. A. Chunikhin, “On Π-properties of finite groups”, *Mat. Sb.* **25(67)**:3 (1949), 321–346; English transl., Amer. Math. Soc. Transl. Ser. 2, vol. 1952, Amer. Math. Soc., Providence, RI 1952, 72.

[9] S. A. Chunikhin, “On existence and conjugateness of subgroups of a finite group”, *Mat. Sb.* **33(75)**:1 (1953), 111–132.

[10] S. A. Chunikhin, “Some trends in the development of the theory of finite groups in recent years”, *Uspekhi Mat. Nauk* **16**:4(100) (1961), 31–50; English transl., *Russian Math. Surveys* **16**:4 (1961), 29–46.

[11] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups*, With comput. assistance from J. G. Thackray, Oxford Univ. Press, Eynsham 1985.

[12] Wenbin Guo and D. O. Revin, “Pronormality and submaximal $\mathfrak{X}$-subgroups in finite groups”, *Commun. Math. Stat.* **6**:3 (2018), 289–317.

[13] E. P. Vdovin and D. O. Revin, "Theorems of Sylow type", *Uspekhi Mat. Nauk* **66**:5(401) (2011), 3–46; English transl., *Russian Math. Surveys* **66**:5 (2011), 829–870.

[14] W. Guo and D. O. Revin, "Conjugacy of maximal and submaximal $\mathfrak{X}$-subgroups", *Algebra i Logika* **57**:3 (2018), 261–278; English transl., *Algebra and Logic* **57**:3 (2018), 169–181.

[15] K. Doerk and T. O. Hawkes, *Finite soluble groups*, De Gruyter Exp. Math., vol. 4, Walter de Gruyter & Co., Berlin 1992.

[16] E. P. Vdovin, N. Ch. Manzaeva, and D. O. Revin, "On the heritability of the Sylow π-theorem by subgroups", *Mat. Sb.* **211**:3 (2020), 3–31; English transl., *Sb. Math.* **211**:3 (2020), 309–335.

[17] S. Lang, *Algebra*, Addison-Wesley Publishing Co., Inc., Reading, MA 1965; Russian transl., Mir, Moscow 1968.

[18] J. N. Bray, D. F. Holt, and C. M. Roney-Dougal, *The maximal subgroups of the low-dimensional finite classical groups*, London Math. Soc. Lecture Note Ser., vol. 407, Cambridge Univ. Press, Cambridge 2013.

[19] M. Suzuki, *Group theory* I, Transl. from the Japan., Grundlehren Math. Wiss., vol. 247, Springer-Verlag, Berlin–New York 1982.

[20] M. Suzuki, *Group theory* II, Transl. from the Japan., Grundlehren Math. Wiss., vol. 248, Springer-Verlag, New York 1986.

[21] P. Hall, "Theorems like Sylow's", *Proc. London Math. Soc.* (3) **6**:2 (1956), 286–304.

[22] B. Huppert, *Endliche Gruppen* I, Grundlehren Math. Wiss., vol. 134, Springer-Verlag, Berlin–New York 1967.

[23] D. O. Revin and E. P. Vdovin, "On the number of classes of conjugate Hall subgroups in finite simple groups", *J. Algebra* **324**:12 (2010), 3614–3652.

[24] V. A. Vedernikov, "Finite groups with Hall π-subgroups", *Mat. Sb.* **203**:3 (2012), 23–48; English transl., *Sb. Math.* **203**:3 (2012), 326–350.

[25] A. A. Buturlakin and A. P. Khramova, "A criterion for the existence of a solvable π-Hall subgroup in a finite group", *Comm. Algebra* **48**:3 (2020), 1305–1313.

Wen B. Guo
School of Science, Hainan University,
Haikou, Hainan, P. R. China;
Department of Mathematics,
University of Science and Technology of China,
Hefei, P. R. China
E-mail: `wguo@ustc.edu.cn`

Received 29/OCT/21
30/JAN/22
Translated by A. R. ALIMOV

Danila O. Revin
Sobolev Institute of Mathematics
Siberian Branch of the Russian Academy
of Sciences, Novosibirsk;
Novosibirsk State University;
Krasovskii Institute of Mathematics and
Mechanics,
Ural Branch of the Russian Academy of Sciences,
Ekaterinburg, Russia
E-mail: `revin@math.nsc.ru`